NPS-53-89-003

NAVAL POSTGRADUATE SCHOOL

Monterey, California

ON SINGULAR VALUES OF HANKEL
OPERATORS OF FINITE RANK

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November 1988

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Prepared for: Naval Postgraduate School and the National Science Foundation, Washington D.C. 20550

NAVAL POSTGRADUATE SCHOOL
Department of Mathematics

This report was prepared in conjunction with research conducted for the National Science Foundation and for the Naval Postgraduate School Research Council and funded by the Naval Postgraduate School Research Council.

Prepared by:

REPORT DOCUMENTATION PAGE

EPORT SECURITY CLASSIFICATION	1b RESTRICTIVE MARKINGS
ECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited
DECLASSIFICATION/DOWNGRADING SCHEDULE	
RFORMING ORGANIZATION REPORT NUMBER(S) NPS-53-89-003	5 MONITORING ORGANIZATION REPORT NUMBER(S) NPS-53-89-003
IAME OF PERFORMING ORGANIZATION al Postgraduate School · 6b OFFICE SYMBOL (If applicable) 53	7a NAME OF MONITORING ORGANIZATION Naval Postgraduate School and the National Science Foundation
DDRESS (City, State, and ZIP Code) nterey, CA 93943	7b ADDRESS (City, State, and ZIP Code) Monterey, CA 93943 and Washington, D.C.
IAME OF FUNDING/SPONSORING ORGANIZATION val Postgraduate School · 8b OFFICE SYMBOL (If applicable) 53	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER O&MN, Direct funding
DDRESS (City, State, and ZIP Code) nterey, CA 93943	10 SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO · PROJECT NO · TASK NO · WORK UNIT ACCESSION NO

ITLE (Include Security Classification)
 Singular Values of Hankel Operators of Finite Rank

ERSONAL AUTHOR(S)
William B. Gragg and L. Reichel

TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
hnical Report	FROM 9/1/87 TO 11/30/87	1 November 1988	19

UPPLEMENTARY NOTATION

COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Hankel operator, singular values, generalized Takagi singular value problem, generalized eigenvalue problem, Lanczos iterations

ABSTRACT (Continue on reverse if necessary and identify by block number)

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DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
NAME OF RESPONSIBLE INDIVIDUAL William Gragg	22b TELEPHONE (Include Area Code) (408) 646-2194 · 22c OFFICE SYMBOL 53Gr

FORM 1473, 84 MAR · 83 APR edition may be used until exhausted · All other editions are obsolete

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

On singular values of Hankel operators of finite rank

W. B. Gragg† and L. Reichel ‡

Abstract

Let H be a Hankel operator defined by its symbol $\rho = \pi/\chi$ where χ is a monic polynomial of degree n and π is a polynomial of degree less than n. Then H has rank n. We derive a generalized Takagi singular value problem defined by two $n \times n$ matrices, such that its n generalized Takagi singular values are the positive singular values of H. If ρ is real, then the generalized Takagi singular value problem reduces to a generalized symmetric eigenvalue problem. The computations can be carried out so that the Lanczos method applied to the latter problem requires only $O(n \log n)$ arithmetic operations for each iteration. If π and χ are given in power form, then the elements of all $n \times n$ matrices required can be determined in $O(n^2)$ arithmetic operations.

This work was supported by NSF under Grant DMS–870416, The Foundation Research Program of the Naval Postgraduate School, and by Bergen Scientific Centre.

Keywords

Hankel operator, singular values, generalized Takagi singular value problem, generalized eigenvalue problem, Lanczos iterations

November 1987

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1. Introduction

Let $H = [\eta_{j+k}]_{j,k=0}^{\infty}$ be a be a Hankel operator defined by its rational symbol $\rho = \pi/\chi$, where

$$\pi(\lambda) := \sum_{j=0}^{n-1} \pi_j \lambda^j \quad \text{and} \quad \chi(\lambda) := \sum_{j=0}^{n} \chi_j \lambda^j, \; \chi_n = 1. \tag{1.1}$$

We assume that π and χ have no common zeros. The elements η_j of H are then given by

$$\rho(\lambda) = \frac{\pi(\lambda)}{\chi(\lambda)} = \sum_{j=0}^{\infty} \eta_j \lambda^{-j-1}. \tag{1.2}$$

In order to simplify our presentation, we assume that the zeros $\{\lambda_k\}_{k=1}^{n}$ of χ are distinct. How our formulas need to be modified in order to remove this assumption is discussed in Remark 1.1 below. Hence ρ has a partial fraction decomposition

$$\rho(\lambda) =: \sum_{k=1}^{n} \frac{\alpha_k}{\lambda - \lambda_k}. \tag{1.3}$$

Expansion of the right hand side of (1.3) into a geometric series, and comparison with (1.2), yields

$$\eta_j = \sum_{k=1}^{n} \alpha_k \lambda_k^j. \tag{1.4}$$

We now express (1.4) in matrix form. Let

$$A := diag[\alpha_1, \alpha_2, \dots \alpha_n] \in \mathbb{C}^{n \times n}, \tag{1.5}$$

$$\Lambda := diag[\lambda_1, \lambda_2, \dots \lambda_n] \in \mathbb{C}^{n \times n}, \tag{1.6}$$

and introduce the Vandermonde matrix

$$V_0 := [\lambda_{k+1}^j]_{j,k=0}^{n-1} \in \mathbb{C}^{n \times n}. \tag{1.7}$$

Define

$$V := [V_j]_{j=0}^{\infty} \in \mathbb{C}^{\infty \times n}, \tag{1.8}$$

where

$$V_j := V_0 \Lambda^{jn}, \; j \geq 1. \tag{1.9}$$

Then (1.4) can be written as

$$H = V A V^T. \tag{1.10}$$

Let l^2 denote the vector space $\mathbb{C}^{\infty}$ equipped with the Euclidean norm.

Proposition 1.1. $H: l^2 \to l^2$ bounded $\Leftrightarrow |\lambda_k| < 1$ for $1 \le k \le n$.

Proof. The proposition holds independent of the multiplicity of the λ_k. In the present proof we assume that the λ_k are distinct. The proof for confluent λ_k is commented on in Remark 1.1.

Let $e_1 = [\varepsilon_j]_{j=0}^{\infty} \epsilon\, \mathbb{C}^{\infty}$ be the axis vector with $\varepsilon_0 = 1$. Then

$$h = [\eta_j]_{j=0}^{\infty} := He_1 \in l^2 \Rightarrow \eta_j \to 0 \text{ as } j \to \infty \Rightarrow$$

$$|\lambda_k| < 1 \text{ for } 1 \le k \le n,$$

where the last implication follows from (1.4).

Conversely, assume that $|\lambda_k| < 1$ for $1 \le k \le n$. Then by (1.8) - (1.10) we obtain

$$\begin{aligned}\|H\|_2 \le \|A\|_2 \|V\|_2^2 &\le \|A\|_2 \|V_0\|_2^2 \|\sum_{j=0}^{\infty} (\Lambda^H \Lambda)^{nj}\|_2^2 \\ &= \|A\|_2 \|V_0\|_2^2 \|(I - (\Lambda^H \Lambda)^n)^{-1}\|_2^2. \quad \blacksquare\end{aligned}$$

We assume henceforth that $|\lambda_k| < 1$ for $1 \le k \le n$. Introduce

$$U := V\, V_0^{-1}, \tag{1.11}$$

$$H_0 := V_0 A V_0^T. \tag{1.12}$$

Then H_0 has rank n. We note , by comparing (1.12) with (1.10), that H_0 is the leading principal $n \times n$ submatrix of H. From (1.10) - (1.12) it follows that

$$H = UH_0U^T. \tag{1.13}$$

The leading $n \times n$ submatrix of U is I_n, the $n \times n$ identity matrix. U therefore is of rank n and can be factored

$$U = QR, \quad Q \,\epsilon\, \mathbb{C}^{\infty \times n}, \quad R \,\epsilon\, \mathbb{C}^{n \times n},$$

where $Q^H Q = I_n$ and R is a nonsingular right triangular matrix. We obtain

$$\sigma_+(H) = \sigma_+(QRH_0R^TQ^T) = \sigma(RH_0R^T), \tag{1.14}$$

where σ denotes the set of singular values and σ_+ denotes the subset of the positive ones.

The $n \times n$ matrix RH_0R^T is complex symmetric. Takagi [Ta1], [Ta2] showed the existence of a complex symmetric singular value decomposition

$$RH_0R^T = W\Sigma W^T,\ W \epsilon\, \mathbb{C}^{n \times n},\ \Sigma = diag[\sigma_1, \sigma_2, \dots \sigma_n], \tag{1.15}$$

where $W^H W = I_n$ and $\sigma_j > 0$ are the singular values of RH_0R^T. In Section 2 we present an elementary proof of the existence of this decomposition. Let $W = [w_1, w_2, \dots w_n]$, $w_j \,\epsilon\, \mathbb{C}^n$. Then (1.15) can be written as the Takagi singular value problem

$$RH_0R^T\overline{w}_j = w_j\sigma_j, \quad w_j^H w_k = \delta_{jk}, \quad 1 \le j, k \le n, \tag{1.16}$$

where the bar denotes complex conjugation and δ_{jk} is Kronecker's δ function. The problems (1.15) - (1.16) could be solved by the algorithm described in [BGG], but this would require RH_0R^T to be explicitly computed. In order to avoid these matrix multiplications we let $v_j := R^H w_j$ and obtain from (1.16) the *generalized Takagi singular value problem*

$$H_0\overline{v}_j = (R^H R)^{-1} v_j \sigma_j, \quad v_j^H (R^H R)^{-1} v_k = \delta_{jk}, \ 1 \leq j, k \leq n. \tag{1.17}$$

The solution of (1.17) requires $(R^H R)^{-1}$ to be known. In Section 3 we show that

$$(R^H R)^{-1} = I - B_0 B_0^H, \tag{1.18}$$

where $B_0 \in \mathbb{C}^{n\times n}$ is a triangular Toeplitz matrix. The elements of B_0 and H_0 can be determined from the coefficients of π and χ in $O(n \log n)$ arithmetic operations by the fast Fourier transform (FFT) method. This is demonstrated in Section 4. Section 5 shows that

$$R^H R = \overline{T_1 M_0 T_1^H}, \quad T_1, M_0 \in \mathbb{C}^{n\times n}, \tag{1.19}$$

where T_1 and M_0 are Toeplitz matrices, and describes a numerical scheme for the computation of this factorization from (1.16) in $O(n^2)$ arithmetic operations. We also present a Hermitian factorization of $R^H R$ into $n \times n$ triangular matrices.

The factorization (1.19) may be of interest for the numerical solution of (1.17). Assume that the coefficients of π and χ are real valued. Then H_0, $(R^H R)^{-1} \in \mathbb{R}^{n\times n}$, and (1.17) reduces to a generalized symmetric eigenvalue problem. The Lanczos method ([Pa, Section 15.11], [ER]) would appear suitable for solving this eigenproblem for the following reason. Let $C \in \mathbb{C}^{n\times n}$ be a Hankel or Toeplitz matrix and let $v \in \mathbb{C}^n$ be arbitrary. It is well known that Cv can be computed in $O(n \log n)$ arithmetic operations using FFTs. Hence $H_0 v$, $(R^H R)^{-1} v$ and $(R^H R)v$ can be computed in $O(n \log n)$ arithmetic operations, where we use (1.18) - (1.19). Each iteration of the Lanczos algorithm given in [Pa, p.324] therefore requires only $O(n \log n)$ arithmetic operations.

The computation of singular values of H is important in Hankel norm approximation problems of systems theory, such as the model reduction problem [Gl]. The approximation of functions by the Carathéodory - Fejér method yields another application [Gu], [Tr].

Other methods for reducing the singular value problem for H to a finite dimensional one have been described by Kung and Gutknecht [Gu] and Young [Yo]. These methods, however, do not preserve symmetry. Moreover, Young's approach requires generally $O(n^3)$ arithmetic operations to compute the matrices required.

Remark 1.1. Formulas (1.3) - (1.8) and the proof of Proposition 1.1 require distinct λ_k. This restriction can be removed. Assume first that $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. Then (1.3) - (1.4) have to be replaced by

$$\rho(\lambda) =: \sum_{k=1}^{n} \frac{\alpha_k}{(\lambda - \lambda_1)^k}, \tag{1.3'}$$

$$\eta_j = \sum_{k=1}^{n} \frac{\alpha_k}{\lambda^k} \Big[\sum_{j=o}^{\infty} (\frac{\lambda_1}{\lambda})^j\Big]^k. \tag{1.4'}$$

In (1.5) A has to be substituted by the upper triangular Hankel matrix

$$A = [\alpha_{j+k+1}]_{j,k=0}^{n-1} \in \mathbb{C}^{n\times n}; \quad \alpha_p := 0, \quad p > n.$$

The matrix Λ in (1.6) has to be replaced by the Jordan matrix with all diagonal elements equal to λ_1 and all superdiagonal elements equal to one. The matrix V_0 in (1.7) need be replaced by the confluent Vandermonde matrix. For instance, we obtain for $n = 3$

$$\Lambda = \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}, \qquad V_0 = \begin{bmatrix} 1 & & \\ \lambda_1 & 1 & \\ \lambda_1^2 & 2\lambda_1 & 1 \end{bmatrix}.$$

With A, Λ and V_0 modified as described, we define V_j and V by (1.8) - (1.9), U by (1.11) and H_0 by (1.12). Then (1.10) and (1.13) hold and H_0 is the leading principal $n \times n$ submatrix of H. Also (1.14) - (1.19) remain valid. Proposition 1.1 can be shown by replacing (1.4) by (1.4′), and by bounding the sum

$$\| \sum_{j=0}^{\infty} (\Lambda^H \Lambda)^{nj} \|_2^2$$

where Λ now is a Jordan matrix. This sum is bounded if $|\lambda_1| < 1$, and the proposition remains valid.

In general, when the λ_k are of arbitrary multiplicity, A in (1.5) has to be replaced by a block diagonal matrix, where each block is an upper triangular Hankel matrix. The blocks are of the same sizes as the multiplicities of the λ_k, and the number of blocks equals the number of distinct λ_k. Λ in (1.6) is replaced by a Jordan matrix with Jordan boxes of the same sizes as the multiplicities of the λ_k, and the number of boxes equal to the number of distinct λ_k. V_0 in (1.7) is replaced by an appropriate confluent Vandermonde matrix. With these changes (1.10) - (1.19) are valid, and so is Proposition 1.1. We omit the details since the numerical computations are independent of the multiplicity of the λ_k. ∎

2. The Symmetric Singular Value Decomposition

In this section we present an elementary proof of Takagi's theorem, i.e. we show the existence of a symmetric singular value decomposition of a complex symmetric matrix. Let $C = C^T \in \mathbb{C}^{n\times n}$, and define $A, B \in \mathbb{R}^{n\times n}$ by $C := A + iB$, $i := \sqrt{-1}$. Then $A = A^T$ and $B = B^T$, so the matrix

$$\tilde{C} := \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

is real and symmetric. Let $\{\sigma_j\}_{j=1}^r$ be the *positive* eigenvalues of $\tilde{C}$ and form

$$\Sigma := diag[\sigma_1, \sigma_2, ..., \sigma_r].$$

Let

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Sigma \tag{2.1}$$

with

$$U, V \in \mathbb{R}^{n\times r}$$

and

$$U^T U + V^T V = I_r.$$

Write (2.1) as

$$\begin{cases} AU + BV = U\Sigma \\ BU - AV = V\Sigma \end{cases} \tag{2.2}$$

and note that (2.2) also can be written as

$$\begin{cases} AV + B(-U) = V(-\Sigma) \\ BV - A(-U) = (-U)(-\Sigma), \end{cases}$$

i.e.

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} V \\ -U \end{bmatrix} = \begin{bmatrix} V \\ -U \end{bmatrix} (-\Sigma) \tag{2.3}$$

with

$$V^T V + (-U)^T(-U) = I_r.$$

Hence $\tilde{C}$ has at least r negative eigenvalues. We could also have let σ_j be the negative eigenvalues of $\tilde{C}$ and then (2.3) would have given us positive ones. We therefore may assume that $\pm\sigma_1, \pm\sigma_2, ..., \pm\sigma_r$ are all the nonzero eigenvalues of $\tilde{C}$.

Since eigenvectors associated with distinct eigenvalues of a real symmetric matrix are orthogonal, we have

$$0 = [V^T, -U^T] \begin{bmatrix} U \\ V \end{bmatrix} = V^T U - U^T V.$$

The spectral resolution of $\tilde{C}$ is thus

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} = \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} \Sigma & \\ & -\Sigma \end{bmatrix} \begin{bmatrix} U^T & V^T \\ V^T & -U^T \end{bmatrix},$$

which yields

$$\begin{cases} A = U\Sigma U^T - V\Sigma V^T \\ B = V\Sigma U^T + U\Sigma V^T. \end{cases}$$

Therefore

$$\begin{aligned} C = A + iB &= U\Sigma U^T - V\Sigma V^T + i(V\Sigma U^T + U\Sigma V^T) \\ &= (U + iV)\Sigma(U^T + iV^T) = W\Sigma W^T = \sum_{k=1}^{r} \sigma_k w w_k^T, \end{aligned}$$

where

$$U + iV =: W = [w_1, w_2, ..., w_r], \; w_k \epsilon \mathbb{C}^n.$$

Moreover

$$W^H W = (U^T - iV^T)(U + iV) = (U^T U + V^T V) + i(U^T V - V^T U) = I_r.$$

If $r < n$ then one may replace Σ by

$$\Sigma_0 := diag[\sigma_1, \sigma_2, ..., \sigma_r, 0, ..., 0] \; \epsilon \; \mathbb{R}^{n \times n}$$

and W by

$$W_0 := [w_1, w_2, ..., w_r, w_{r+1}, ... w_n] \; \epsilon \; \mathbb{C}^{n \times n},$$

where $w_{r+1}, ..., w_n \epsilon \; \mathbb{C}^n$ are chosen so that $W_0^H W_0 = I_n$. ∎

3. A Simple Expression for $(R^H R)^{-1}$

In this section we derive (1.18). Introduce the Frobenius matrix

$$F := [e_2, e_3, ..., e_n, -f] \in \mathbb{C}^{n \times n},$$

where

$$e_j := [\delta_{1j}, \delta_{2j}, ..., \delta_{nj}]^T \in \mathbb{R}^n, \quad 2 \leq j \leq n, \qquad (3.1)$$
$$f := [\chi_0, \chi_1, ..., \chi_{n-1}]^T \in \mathbb{C}^n.$$

Then F is the companion matrix of χ and

$$F^T V_0 = V_0 \Lambda. \qquad (3.2)$$

Throughout this section V_0 and Λ are defined by (1.6) - (1.7) if the λ_k are distinct. For confluent λ_k we modify V_0 and Λ according to Remark 1.1. The following lemma shows that

$$G := \overline{R^H R} \qquad (3.3)$$

satisfies a Stein equation. This will enable us to obtain a simple expression for G^{-1} by an application of the Sherman-Morrison-Woodbury formula.

Lemma 3.1. G is the unique solution of the Stein equation

$$X - F^n X F^{nH} = I_n, \quad X \in \mathbb{C}^{n \times n}. \qquad (3.4)$$

Proof. By (1.8), (1.9) and (1.11) we obtain

$$R^H R = U^H U = \sum_{k=0}^{\infty} V_0^{-H} (\Lambda^{nk})^H V_0^H V_0 \Lambda^{nk} V_0^{-1}, \qquad (3.5)$$

and (3.2) yields now

$$G = \sum_{k=0}^{\infty} F^{nk} (F^{nk})^H. \qquad (3.6)$$

The series in (3.5) - (3.6) converge because $|\lambda_k| < 1$ for all k. Substitution of (3.6) into (3.4) shows that G solves (3.4). The unicity follows from $|\lambda_k| < 1$ for all k. The latter can be seen by a similarity transform of F^n to Schur triangular form. ∎

Introduce the cyclic downshift operator in $\mathbb{C}^{2n}$

$$E := [e_2, e_3 ..., e_n, e_1] \in \mathbb{C}^{2n \times 2n},$$

where

$$e_j := [\delta_{1j}, \delta_{2j}, ..., \delta_{2n,j}]^T \in \mathbb{R}^{2n}. \qquad (3.7)$$

Let

$$t := [\chi_0, \chi_1, ..., \chi_n, 0, 0, ..., 0]^T \in \mathbb{C}^{2n},$$

and define the Toeplitz matrix T of parallelogram form

$$T := [t, Et, E^2 t, \ldots, E^{n-1} t] \in \mathbb{C}^{2n \times n}. \tag{3.8}$$

Let T_0 be the leading $n \times n$ submatrix of T, and let T_1 be the trailing $n \times n$ submatrix of T. Then T_0 is a left triangular Toeplitz matrix, and T_1 is a unit right triangular Toeplitz matrix.

Example 3.1. Let $n = 3$. Then

$$T = \begin{bmatrix} \chi_0 & & \\ \chi_1 & \chi_0 & \\ \chi_2 & \chi_2 & \chi_0 \\ \chi_3 & \chi_2 & \chi_1 \\ & \chi_3 & \chi_2 \\ & & \chi_3 \end{bmatrix}, \qquad T_0 = \begin{bmatrix} \chi_0 & & \\ \chi_1 & \chi_0 & \\ \chi_2 & \chi_1 & \chi_0 \end{bmatrix}, \qquad T_1 = \begin{bmatrix} \chi_3 & \chi_2 & \chi_1 \\ & \chi_3 & \chi_2 \\ & & \chi_3 \end{bmatrix},$$

where we note that $\chi_3 = 1$. ∎

Lemma 3.2. Let T_0 and T_1 be defined as above. Then

$$T_0^H T_0 + T_1^H T_1 = T_0 T_0^H + T_1 T_1^H. \tag{3.9}$$

Proof. Let $N := T^H T = T_0^H T_0 + T_1^H T_1$. We first show that N is a Toeplitz matrix. Let e_j be defined by (3.1). Then by (3.8) we have for $1 \le j, k \le n$,

$$e_j^T \; N e_k = e_j T^H T \; e_k = t^H (E^H)^{j-1} E^{k-1} t = t^H E^{k-j} t,$$

where we have used that $E^H = E^{-1}$. We next define the reversal matrix

$$J := [e_n, e_{n-1}, \ldots, e_1] \in \mathbb{R}^{n \times n}.$$

Toeplitz matrices are counter symmetric, i.e. $N = J N^T J$. Using that N is counter symmetric and Hermitian yields

$$\begin{aligned} T_0^H T_0 + T_1^H T_1 &= N = J N^T J = J \overline{N} J = J (T_0^T \overline{T_0} + T_1^T \overline{T_1}) J \\ &= J T_0^T J \cdot J \overline{T_0} J + J T_1^T J \cdot J \overline{T_1} J = T_0 T_0^H + T_1 T_1^H. \quad ∎ \end{aligned}$$

The next lemma presents a Gaussian factorization of F^n in terms of T_0 and T_1. This will be used together with Lemma 3.1 to express G^{-1} in terms of T_0 and T_1.

Lemma 3.3.

$$F^n = -T_0 T_1^{-1}. \tag{3.10}$$

Proof. We first show that

$$[T_0^T, T_1^T] \begin{bmatrix} V_0 \\ \\ V_0 \Lambda^n \end{bmatrix} = 0. \tag{3.11}$$

Let e_j be defined by (3.7) and assume for the moment that the λ_k are distinct. Then

$$e_j^T \; [T_0^T, T_1^T] \begin{bmatrix} V_0 \\ \\ V_0 \Lambda^n \end{bmatrix} e_k = \chi(\lambda_k) \lambda_k^{j-1} \tag{3.12}$$

and the right hand side vanishes for $1 \leq j,k \leq n$. If the λ_k are confluent, then the right hand side expression of (3.12) contains derivatives of $\chi(\lambda)$ evaluated at λ_k. The right hand side of (3.12), however, still vanishes and (3.11) holds.

We now write (3.11) as

$$T_0^T V_0 + T_1^T V_0 \Lambda^n = 0$$

and apply (3.2). This shows (3.10). ∎

We are now in a position to show (1.18). By (3.4) G satisfies

$$G = I + F^n G\ F^{nH}$$

and an application of the Sherman-Morrison-Woodbury formula yields

$$G^{-1} = (I + F^n G\ F^{nH})^{-1} = I - F^n (G^{-1} + F^{nH} F^n)^{-1} F^{nH}. \tag{3.13}$$

We now determine an expression for

$$Y := I - G^{-1}. \tag{3.14}$$

Substitute Y and (3.10) into (3.13) to obtain

$$Y = T_0 (T_0^H T_0 + T_1^H T_1 - T_1^H Y T_1)^{-1} T_0^H. \tag{3.15}$$

In order to determine a simple expression for Y from (3.15) we need the following observation, which is also central to Section 4. T_0 and T_1^{-H} are both left triangular $n \times n$ Toeplitz matrices. Multiplication of T_0 with T_1^{-H} can be identified with polynomial multiplication, see [He1, Section 1.3] and Section 4. Since multiplication of polynomials commutes, we obtain

$$T_0 T_1^{-H} = T_1^{-H} T_0. \tag{3.16}$$

From the correspondence between polynomials and left triangular Toeplitz matrices it also follows that $T_0 T_1^{-H}$ is a left triangular Toeplitz matrix.

Lemma 3.4. Equation (3.15) has the unique solution

$$Y = T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 T_1^{-H} T_1^{-1} T_0^H. \tag{3.17}$$

Proof. Unicity follows from (3.14) and that (3.4) has a unique solution. From (3.16) we obtain

$$T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 T_1^{-H} T_1^{-1} T_0^H. \tag{3.18}$$

Now substitute

$$Y = T_1^{-H} T_0 T_0^H T_1^{-1}$$

into (3.15). We obtain

$$T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 (T_0^H T_0 + T_1^H T_1 - T_0 T_0^H)^{-1} T_0^H. \tag{3.19}$$

An application of (3.9) reduces (3.19) to (3.18). The latter has already been shown to be valid. Therefore (3.17) solves (3.15). ∎

Let

$$B_0 := \overline{T}_0 T_1^{-T} = T_1^{-T} \overline{T}_0. \tag{3.20}$$

Then B_0 is a left triangular $n \times n$ Toeplitz matrix. By (3.14) and (3.17)

$$G^{-1} = I - \overline{B}_0 B_0^T = I - B_0^T \overline{B}_0.$$

From (3.3) it now follows that

$$(R^H R)^{-1} = I - B_0 B_0^H. \tag{3.21}$$

4. Computation of H_0 and B_0

We summarize some results in [He 1, Section 1.3] and [He 2, Section 13.9] in order to show that the elements of H_0 and B_0 can be computed in $O(n \log n)$ arithmetic operations from the coefficients χ_j of χ and π_j of π, see (1.1). To a polynomial or power series

$$\varsigma(\lambda) := \sum_{j=0}^{n-1} \varsigma_j \lambda^j + O(\lambda^n)$$

we associate the left triangular $n \times n$ Toeplitz matrix

$$Z = [\varsigma_{j-k}]_{j,k=0}^{n-1}, \qquad \varsigma_j = 0 \text{ for } j < 0,$$

and we write $\varsigma \to Z$. If $\xi(\lambda)$ is a polynomial and X a left triangular $n \times n$ Toeplitz matrix such that $\xi \to X$, then it is easily seen that $\varsigma\xi \to ZX$. In particular, ZX is a left triangular $n \times n$ Toeplitz matrix. From $\xi\varsigma = \varsigma\xi$ and $\xi\varsigma \to XZ$ is follows that $ZX = XZ$.

Assume that $\varsigma_0 \neq 0$ and let $1/\varsigma \to Z'$. Then $1/\varsigma \cdot \varsigma \to I$, $Z'Z$ and ZZ'. We obtain $Z' = Z^{-1}$ and therefore Z^{-1} is a left triangular Toeplitz matrix.

Example 4.1. We have $\chi \to T_0$. Let

$$\tilde{\chi}(\lambda) := \lambda^n \; \overline{\chi}(1/\lambda) = \sum_{j=0}^{n} \overline{\chi}_{n-j} \lambda^j. \tag{4.1}$$

Then $\tilde{\chi} \to T_1^H$ and the Blaschke product

$$\frac{\chi}{\tilde{\chi}} \to T_0 T_1^{-H} = \overline{B}_0. \tag{4.2}$$

■

Now let $\xi(\lambda)$ and $\varsigma(\lambda)$ be arbitrary polynomials such that $\varsigma(0) \neq 0$. Henrici [He2, Theorem 13.9e] shows that the first n coefficients in the MacLaurin expansion of $\xi(\lambda)/\varsigma(\lambda)$ can be computed in $O(n \log n)$ multiplications. The proof uses FFT. It is easily seen that the number of additions also is $O(n \log n)$.

From $\chi_n = 1$ and (4.1) we obtain $\tilde{\chi}(0) \neq 0$. Hence, the first n terms in the MacLaurin expansion of $\chi/\tilde{\chi}$ can be computed in $O(n \log n)$ arithmetic operations. By (4.2) therefore $\overline{T_0 T_1^{-H}} = B_0$ can be computed in $O(n \log n)$ arithmetic operations.

Because $\lambda^n \chi(1/\lambda) \neq 0$ for $\lambda = 0$, we can compute the first n terms in the MacLaurin expansion of

$$\frac{\lambda^n \pi(1/\lambda)}{\lambda^n \chi(1/\lambda)} = \sum_{j=0}^{n-1} \eta_j \lambda^{j+1} + O(\lambda^n)$$

in $O(n \log n)$ arithmetic operations. This shows that H_0 can be computed in $O(n \log n)$ arithmetic operations.

5. A Factorization of $R^H R$

It follows from (3.3) and (3.20) - (3.21) that

$$G^{-1} = \overline{(R^H R)}^{-1} = I - \overline{B_0\ B_0^H} = I - T_1^{-H} T_0 T_0^H T_1^{-1}, \tag{4.1}$$

and therefore

$$T_1^H G^{-1} T_1 = T_1^H T_1 - T_0 T_0^H =: M_0^{-1}. \tag{4.2}$$

The expression defining M_0^{-1} is a Gohberg-Semencul formula for the inverse of an $n \times n$ Toeplitz matrix, see, e.g., [Io, Theorem 18.2, p. 152]. We denote this Toeplitz matrix by M_0. From the left hand expression of (4.2) and the nonsingularity of T_1 and R it follows that M_0 is Hermitian and positive definite. The desired factorization of $R^H R$ is

$$R^H R = \overline{T_1\ M_0\ T_1^H}.$$

We will now show how M_0 can be computed. The computation involves running the Levinson algorithm backwards.

Consider the related Gohberg-Semencul formula, see, e.g., [Io, Theorem 18.1, p. 148] or [AG],

$$M_1^{-1} = \begin{bmatrix} \chi_n & \chi_{n-1} & \cdots & \chi_0 \\ & \ddots & & \vdots \\ & & & \chi_{n-1} \\ & & & \chi_n \end{bmatrix}^H \begin{bmatrix} \chi_n & \chi_{n-1} & \cdots & \chi_0 \\ & \ddots & & \vdots \\ & & & \chi_{n-1} \\ & & & \chi_n \end{bmatrix}$$
$$- \begin{bmatrix} 0 & & & \\ \chi_0 & \ddots & & \\ \chi_1 & & & \\ \vdots & & & \\ \chi_{n-1} & \cdots & \chi_1 & \chi_0 & 0 \end{bmatrix} \begin{bmatrix} 0 & & & \\ \chi_0 & \ddots & & \\ \chi_1 & & & \\ \vdots & & & \\ \chi_{n-1} & \cdots & \chi_1 & \chi_0 & 0 \end{bmatrix}^H \tag{4.3}$$

where the four triangular Toeplitz matrices define the inverse of an $(n+1) \times (n+1)$ Hermitian Toeplitz matrix. Denote this Toeplitz matrix by M_1. Then M_0 is the leading principal $n \times n$ submatrix of M_1, see [Io, Theorems 18.1 - 18.2].

Let $R_1 := [\rho_{jk}]_{j,k=0}^n \ \epsilon\ \mathbb{C}^{(n+1)\times(n+1)}$ be the unit right triangular matrix, and let $D_1 := diag[\delta_0, \delta_1, ..., \delta_n]$ be the diagonal matrix such that

$$R_1^H\ M_1\ R_1 = D_1. \tag{4.4}$$

Given $M_1 = [\mu_{j-k}]_{j,k=0}^{n}$, the matrices R_1 and D_1 can be computed by the Levinson algorithm, and by comparing R_1 with (4.3) one finds that

$$\rho_{jn} = \chi_j,\ 0 \le j \le n \text{ and } \delta_n = \chi_n,$$

see, e.g., [AG]. We now apply the recursion formula in Levinson's algorithm backwards in order to determine R_1 and D_1 from the last column of R_1 and δ_n. Then the recursion formula is used forwards to determine M_0. We will also obtain a Hermitian factorization of $R^H R$ into triangular matrices.

Backward Levinson algorithm

input: $[\rho_{jn}]_{j=0}^{n}, \delta_n$; output: R_1, D_1, Schur parameters $\{\gamma_j\}_{j=1}^{n}$ of M_0;

for $k := n, n-1, n-2, \ldots, 1$ **do**

$\gamma_k := \rho_{ok};\ \rho_{k-1,k-1} := 1;$

for $j := 1, 2, \ldots,$ integer part$(\frac{k}{2})$ **do**

solve for $\rho_{j-1,k-1}$ and $\rho_{k-1-j,k-1}$ the linear system of equations

$$\begin{bmatrix} 1 & \gamma_k \\ \overline{\gamma}_k & 1 \end{bmatrix} \begin{bmatrix} \rho_{j-1,k-1} \\ \overline{\rho}_{k-1-j,k-1} \end{bmatrix} = \begin{bmatrix} \rho_{j,k} \\ \overline{\rho}_{k-j,k} \end{bmatrix};$$

$\delta_{k-1} := (\delta_k/(1-|\gamma_k|))/(1+|\gamma_k|);$

Levinson recursion for computing $M_0 = [\mu_{j-k}]_{j,k=0}^{n-1}$

input: $R_1, D_1, \{\gamma_j\}_{j=1}^{n}$; output: $\{\mu_j\}_{j=0}^{n-1}$;

$\mu_0 := \delta_0;\ \mu_1 := -\delta_0\overline{\gamma}_1;$

for $k := 1, 2, \ldots, n-1$ **do**

$\mu_{k+1} := -\delta_k\overline{\gamma}_{k+1} - \sum_{j=1}^{k} \mu_j \overline{\rho}_{j-1,k};$

Hence M_0, R_1, and D_1 are computed in $O(n^2)$ arithmetic operations from the coefficients of χ. Let R_0 and D_0 denote the $n \times n$ leading principal submatrices of R_1 and D_1 respectively. Similarly to (4.4) we have

$$R_0^H M_0 R_0 = D_0. \tag{4.5}$$

Because M_0 is positive definite, so is D_0. $D_0^{1/2}$ can therefore easily be computed. We obtain from

(4.1) - (4.2) and (4.5), with $\hat{R} := D_0^{1/2} R_0^{-1}$,

$$R^H R = (\hat{R} T_1^H)^T \, \overline{(\hat{R} T_1^H)}. \tag{4.6}$$

The right hand side of (4.6) is a Hermitian factorization into triangular matrices. It can be computed in $O(\,n^2)$ arithmetic operations from the coefficients of χ.

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